You Can't Spell "Fundamental Theorem of Algebra" without F-U-N!

2.7

Quadratics and Complex Numbers

LEARNING GOALS

In this lesson, you will:

- Determine the number and type of zeros of a quadratic function.
- Solve quadratic equations with complex solutions.
- Use the Fundamental Theorem of Algebra.
- Choose an appropriate method to determine zeros of quadratic functions.

KEY TERMS

- imaginary roots
- discriminant
- imaginary zeros
- Fundamental Theorem of Algebra
- double root

I'm sure you've heard these sayings: "That's it! I'm drawing a line in the sand!" or "You've crossed that line a long time ago!" So what is the human fascination with lines and boundaries and the implications if these boundaries are crossed?

You might remember that as a young child, you might have been told to stay within the lines of a drawing when coloring in your coloring book; though young toddlers seem to have a tough time with that motor skill. The implication of coloring *outside* of the lines was not a big deal—unless it just drove you crazy as a kid! However, crossing a line over international flying zones, at sea, or on the ground can have much greater impacts on global situations. For example, in 2009, three Americans were hiking in Iraqi Kurdistan when they inadvertently crossed the Iranian border. Upon doing so, Iranian border guards detained the hikers. A long ordeal of negotiating followed between the U.S. and Iranian governments led to the release of Sarah Shourd on September 2010, and the release of Shane Bauer and Joshua Fattal on September 21, 2011.

These of course are two extremes of crossing lines or boundaries. And of course, it is more than governments and conventions that establish lines and boundaries. People establish certain boundaries and etiquette in social settings. What are some social boundaries people have established? What are the implications when those boundaries are crossed?

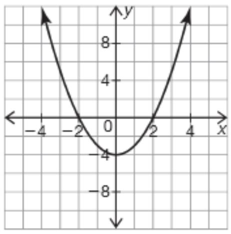
2

PROBLEM 1 X-Axis Intersection Inspection

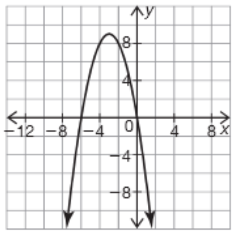
1. Analyze each graph. Identify the x-intercepts, if possible.

Group A

$g(x) = x^2 - 4$

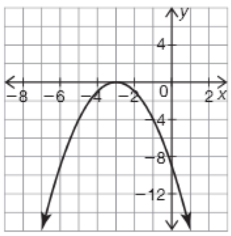


$h(x) = -x^2 - 6x$

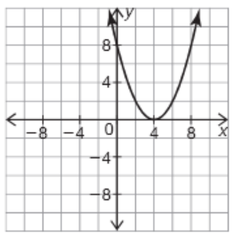


Group B

$m(x) = -x^2 - 6x - 9$

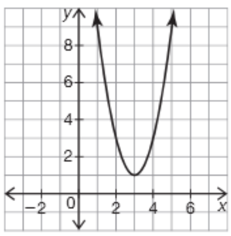


$n(x) = \frac{1}{2}x^2 - 4x + 8$

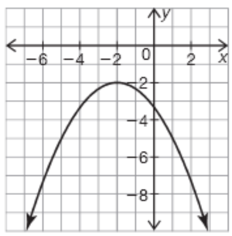


Group C

$p(x) = 2x^2 - 12x + 19$



$q(x) = -\frac{1}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$



The x-intercepts of a quadratic function $f(x)$ are the solutions of the equation $f(x) = 0$.

2. If $f(x)$ is a quadratic function, explain how to determine the number of solutions of $f(x) = 0$ given the graph of $f(x)$.

2

3. Choose one of the two functions from each group in Question 1. Use the Quadratic Formula and what you know about imaginary numbers to solve an equation of the form $f(x) = 0$ for each function you choose.

Group A

$g(x) = x^2 - 4$ $h(x) = -x^2 - 6x$

Group B

$m(x) = -x^2 - 6x - 9$ $n(x) = \frac{1}{2}x^2 - 4x + 8$

Group C

$p(x) = 2x^2 - 12x + 19$

$q(x) = -\frac{1}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$

2

Equations that have imaginary solutions have **imaginary roots**.

4. Consider the three equations you chose to solve in Question 3.
   a. Which of the three equations have imaginary roots?

   b. When you used the Quadratic Formula to solve the equations, at what point did you know the solution was going to include an imaginary number?

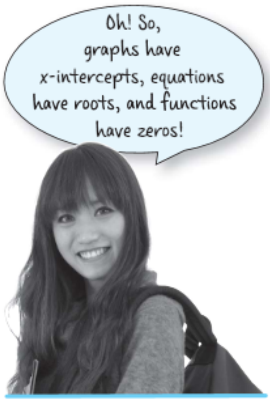
The radicand expression in the Quadratic Formula, $b^2 - 4ac$, is called the **discriminant** because it "discriminates" the number and type of roots of a quadratic equation.

5. Describe how you can tell whether a quadratic equation has real or imaginary roots from the:
   a. discriminant.

   b. graph.

6. Consider the three equations that you did not solve in Question 3. Use the discriminant to determine whether the roots are real or imaginary. Show your work. Then, look at each graph to verify.

Just as equations may have imaginary roots, functions may have *imaginary zeros*. Imaginary zeros are zeros of quadratic functions that do not cross the x-axis. Remember that zeros of a function $f(x)$ are the values of x for which $f(x) = 0$. Zeros, roots, and x-intercepts are all related.



7. Use any method to determine whether each function has real or imaginary zeros. You do not need to calculate the zeros.

   a. $f(x) = -3x^2 + 2x - 1$

   b. $f(x) = -\frac{1}{2}x^2 + x - \frac{1}{2}$

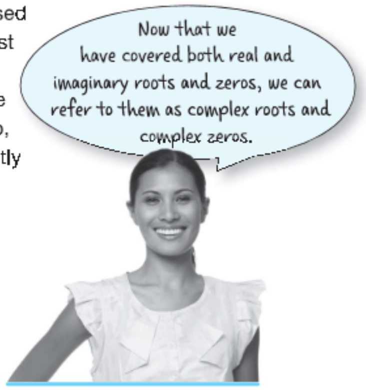
   c. $f(x) = 2x^2 - 5x - 6$

Remember, a quadratic equation is a special type of polynomial equation. The degree of a polynomial equation is the greatest exponent in the polynomial equation.

8. What is the degree of a quadratic equation? a linear equation? a constant equation?

2

The *Fundamental Theorem of Algebra was* first proposed in the early 1600s, but would not be proven until almost two centuries later. The **Fundamental Theorem of Algebra** states that any polynomial equation of degree n must have exactly n complex roots or solutions; also, every polynomial function of degree n must have exactly n complex zeros. However, any root or zero may be a multiple root or zero.



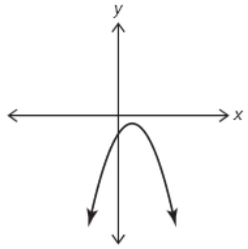
9. Look at the graphs of the functions in Group B of Question 1. Even though the functions intersect the x-axis once, according to the Fundamental Theorem of Algebra, how many complex zeros do these functions have?

If the graph of a quadratic function $f(x)$ has 1 x-intercept, the equation $f(x) = 0$ still has 2 real roots. In this case, the 2 real roots are considered a **double root.**

10. Analyze the discriminants of the quadratic functions in Question 1. What must be true about the discriminant of a quadratic equation that has a double root?

11. Explain why it is not possible for a quadratic equation to have 2 equal imaginary solutions, or a double imaginary root.

12. Circle the function(s) shown that could describe the given graph. Explain your reasoning.



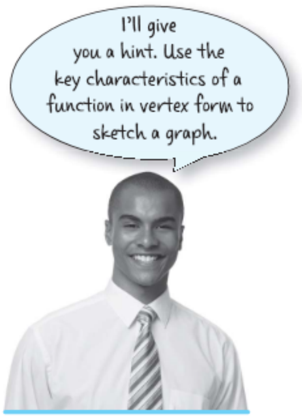
$h(x) = -2x^2 - 3x - 2$ $k(x) = -0.5x^2 + 1.5x + 1$

$t(x) = -\frac{1}{2}x^2 + 3x - \frac{9}{2}$ $w(x) = 2x^2 - 4x - 10$

PROBLEM 2 What Does the Form Tell Me?

1. Consider each quadratic function written in vertex form. Determine whether the zeros of each function are real or imaginary without calculating the zeros. Explain how you know.

a. $f(x) = -2(x - 3)^2 - 4$

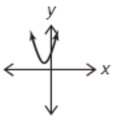
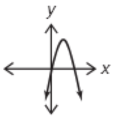
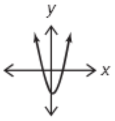
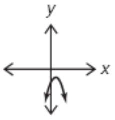
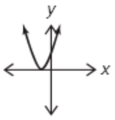
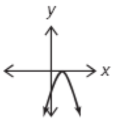


b. $f(x) = 2(x - 3)^2 - 4$

c. $f(x) = -2(x - 3)^2$

2

2. Complete the table to show how you can use the vertex form of a quadratic function to determine whether it has real or imaginary solutions. The first row has been completed for you.

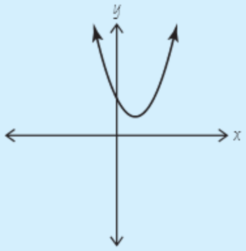
Location of Vertex	Concavity	Sketch	Number of x-Intercepts	Number and Type of Roots	Number and Type of Zeros
Above the x-axis	Up		0	2 imaginary roots	2 imaginary zeros
	Down				
Below the x-axis	Up				
	Down				
On the x-axis	Up				
	Down				

3. Consider the quadratic function $f(x) = \frac{1}{2}(x - 3)(x + 2)$. Describe the type of zeros for this function. How do you know?

4. Tony and Ava each attempt to factor $f(x) = x^2 - 2x + 2$. Analyze their work.

Tony

I sketched the graph $f(x) = x^2 - 2x + 2$ and noticed there are no real zeros. So, $f(x) = x^2 - 2x + 2$ cannot be factored.



Eva

$$x^2 - 2x + 2 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 \pm 2i}{2}$$

$$x = 1 \pm i$$

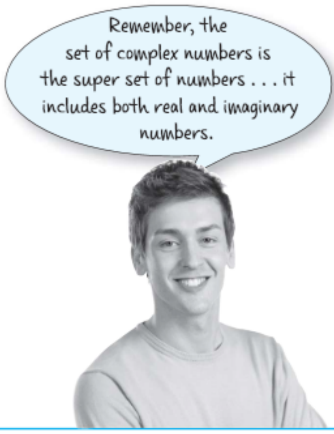
The function in factored form is

$f(x) = [x - (1 + i)][x - (1 - i)]$.

2

a. If you consider the set of real numbers, who's correct? If you consider the set of imaginary numbers, who's correct? Explain your reasoning.

b. Use the distributive property to rewrite Eva's function to verify that the function in factored form is the same as the original function in standard form.



c. Identify the zeros of the quadratic function $f(x)$.

Some functions can be factored over the set of real numbers, while other functions can be factored over the set of imaginary numbers. However, all functions can be factored over the set of complex numbers.

2

5. Analyze each expression.

$x^2 + 4$	$x^2 - 4$	$x^2 + 2x + 5$	$x^2 + 4x - 5$
$-x^2 + x + 12$	$x^2 + 4x - 1$	$-x^2 + 6x - 25$	

a. Sort each expression based on whether it can be factored over the set of real numbers or the set of imaginary numbers.

Complex Factors	
Real Factors	**Imaginary Factors**

b. Factor each expression over the set of complex numbers.

6. Suppose that you know that one zero of a quadratic function $b(x)$ is $2 + 3i$.

a. What is the other zero of the function $b(x)$? Explain how you know.

b. Write the quadratic function $b(x)$ in standard form, using an a-value of 1. Show all your work.

Talk the Talk

The table shown summarizes multiple methods to determine the complex zeros of a quadratic function based specific given information.

	Type of Zeros	Number of Complex Zeros of a Quadratic Function		
		Two Real Zeros		Two Imaginary Zeros
		Two Distinct Zeros	Two Repeated Zeros	Two Distinct Zeros
Given Information	Graph	two x-intercepts	one x-intercept	zero x-intercepts
	Equation in Standard Form: $f(x) = ax^2 + bx + c$	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
	Equation in Vertex Form: $f(x) = a(x - h)^2 + k$	$k > 0$ and $a < 0$ or $k < 0$ and $a > 0$	$k = 0$	$k > 0$ and $a > 0$ or $k < 0$ and $a < 0$
	Equation in Factored Form: $f(x) = a(x - r_1)(x - r_2)$	$r_1 \neq r_2$, r_1 and r_2 are real numbers	$r_1 = r_2$, r_1 and r_2 are real numbers	$r_1 \neq r_2$, r_1 and r_2 are imaginary numbers

Determine the number of zeros and the type of zeros for each quadratic function. Justify your reasoning.

1. $k(x) = -2(x + 3)^2$

2

2. $v(x) = 0.5x^2 - 3x + 10$

3. $c(x) = -\frac{1}{3}x(x - 9)$

4. $p(x) = 5(x - 1)^2 + 6$

Be prepared to share your solutions and methods.